

# Interpolating Dispersionless Integrable System

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## Abstract

We introduce a dispersionless integrable system which interpolates between the dispersionless Kadomtsev–Petviashvili equation and the hyper–CR equation. The interpolating system arises as a symmetry reduction of the anti–self–dual Einstein equations in  $(2, 2)$  signature by a conformal Killing vector whose self–dual derivative is null. It also arises as a special case of the Manakov–Santini integrable system. We discuss the corresponding Einstein–Weyl structures.

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# 1 Introduction

It has been known for more than 20 years that many integrable systems admitting soliton solutions arise as symmetry reductions of anti-self-dual Yang Mills (ASDYM) equations in four dimensions [25]. The Riemann–Hilbert factorisation problem underlies this approach to integrability: it appears in classical solution generating techniques like dressing transformations [18], as well as in the twistor treatment of ASDYM [24].

The dispersionless integrable systems in 2+1 dimensions do not fit into this framework: they do not admit soliton solutions and there is no associated Riemann–Hilbert problem where the corresponding Lie group is finite dimensional. These systems can nevertheless be described in terms of anti-self-duality (ASD) conditions on a four-dimensional conformal structure. In this case the ‘unknown’ in the equations is not a gauge field, but rather a metric (up to scale) on some four-manifold [21]. This makes the dispersionless systems more geometric than their solitonic cousins. This point of view may be of deep significance in description of shock formations: a recent beautiful analysis of Manakov and Santini [16] deduced the gradient catastrophe of the localised solutions to the dKP equation using the inverse scattering transform. It may be however, that this catastrophe is only an artifact of a chosen coordinate system, and the underlying conformal structure is regular, but needs to be covered by more than one coordinate patch. It remains to be seen whether this is indeed the case.

To classify existing 2+1 dispersionless integrable systems, and perhaps discover some new ones one needs to classify the symmetry reductions of the conformal ASD equations. If the Ricci-flat condition is imposed on top of the anti-self-duality, the work of Plebański [22] implies the existence of a local coordinate system  $(X, Y, W, Z)$  and a function  $\Theta$  on an open set  $\mathcal{M} \subset \mathbb{R}^4$  such that any ASD Ricci-flat metric is locally of the form

$$g = 2(dZdY + dWdX - \Theta_{XX}dZ^2 - \Theta_{YY}dW^2 + 2\Theta_{XY}dWdZ), \quad (1.1)$$

where  $\Theta_{XY} = \partial_X \partial_Y \Theta$  etc, and  $\Theta$  satisfies the second heavenly equation

$$\Theta_{ZY} + \Theta_{WX} + \Theta_{XX}\Theta_{YY} - \Theta_{XY}^2 = 0. \quad (1.2)$$

This metric has signature  $(++--)$  but this is precisely what need: Given a non-null symmetry, the conformal structure on a space of orbits will have signature  $(2, 1)$  and will be described by a hyperbolic integrable equation. We are thus led to study (1.1), (1.2) subject to the existence of a conformal Killing vector  $K$

$$\mathcal{L}_K g = cg, \quad g(K, K) \neq 0$$

where  $c$  is some function. The classification of reductions is based on studying the action of  $K$  on the bundle of self-dual two-forms over  $\mathcal{M}$ . Assume that  $\mathcal{M}$  is oriented and recall that given a metric of  $(++--)$  signature on  $\mathcal{M}$ , the Hodge  $*$  operator is an involution on two-forms and induces a decomposition

$$\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$$

of two-forms into self-dual and anti-self-dual components. Moreover there exist real two-dimensional vector bundles  $\mathbb{S}$  and  $\mathbb{S}'$  (called spin bundles) over  $\mathcal{M}$  such that  $T\mathcal{M} \cong \mathbb{S} \otimes \mathbb{S}'$  and  $\Lambda_+^2 \cong \mathbb{S} \odot \mathbb{S}$ . Therefore the self-dual derivative of (the one-form metric dual to)  $K$

$$dK_+ := \frac{1}{2}(dK + *dK)$$

corresponds to a symmetric 2 by 2 matrix  $\phi$  explicitly given by

$$dK_+ = \phi_{AB} \Sigma^{AB}, \quad A, B = 0, 1,$$

where the self-dual two forms<sup>1</sup>  $(\Sigma^{00}, \Sigma^{01}, \Sigma^{11})$  span  $\Lambda_+^2$ . The rank of the matrix  $\phi$  does not depend on the choice of the basis  $\Sigma^{AB}$  and the classification is based on this rank. In the Riemannian signature this can only be 0 or 2 (the former case is called the tri-holomorphic symmetry), but in the  $(++--)$  signature we can also have  $\text{rank}(\phi) = 1$ , in which case  $(dK)_+ \wedge (dK)_+ = 0$  so the self-dual derivative of  $K$  is null.

This classification programme has almost been completed and one aim of this paper is to remove the question mark from the following table summarising the reductions of the second heavenly equation (1.2)

	$c = 0$	$c \neq 0$
$\text{rank}(\phi) = 0$	2+1 Linear wave equation [11, 9]	Hyper-CR equation [3, 4]
$\text{rank}(\phi) = 1$	Dispersionless KP equation [5]	?
$\text{rank}(\phi) = 2$	SU( $\infty$ ) Toda equation [9]	Integrable equation studied in [6]

The case where  $\text{rank}(\phi) = 1$  and  $c \neq 0$  has not yet been investigated, and the resulting integrable system is the subject of the present paper. This system is given by

$$u_y + w_x = 0, \quad u_t + w_y - c(uw_x - wu_x) + buu_x = 0, \quad (1.3)$$

where  $b$  and  $c$  are constants and  $u, w$  are smooth functions of  $(x, y, t)$ . We propose to call (1.3) an interpolating system as it contains two well known dispersionless equations as the limiting cases: Setting  $b = 0, c = -1$  gives the hyper-CR equation [3, 20, 8, 4, 17, 14, 19] and setting  $c = 0, b = 1$  gives the dKP equation. In fact one constant can always be eliminated from (1.3) by redefining the coordinates and it is only the ratio of  $b/c$  which remains<sup>2</sup>. We prefer to keep both constants as it makes various limits more transparent.

In the next Section we shall give the dispersionless Lax pair for (1.3). The Lie group underlying the Lax formulation is  $\text{Diff}(\Sigma^2)$  – an infinite-dimensional group of diffeomorphisms of some two-dimensional manifold  $\Sigma^2$ . The interpolating system corresponds to Lorentzian Einstein–Weyl structures in three dimensions. This gives an intrinsic geometric interpretation without the need of going to four-dimensions. This will be described in Section 3. In Section 4 we shall show that (1.3) is a special case of the Manakov–Santini integrable system [15, 14] and find the Manakov–Santini Einstein–Weyl structure.

The explicit reduction of the second heavenly equation (1.2) to the interpolating system (1.3) will be presented in the Appendix. In particular we shall show that the most general

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<sup>1</sup>The ASD Ricci-flat equations are equivalent to

$$d\Sigma^{AB} = 0, \quad \Sigma^{(AB} \wedge \Sigma^{CD)} = 0, \quad A, B, C, D = 0, 1$$

and the second heavenly equation (1.2) arises by using the Darboux theorem to introduce coordinates such that

$$\Sigma^{00} = dW \wedge dZ, \quad \Sigma^{01} = dW \wedge dX + dZ \wedge dY,$$

and deducing the existence of the function  $\Theta$  from the remaining conditions on  $\Sigma^{AB}$ .

<sup>2</sup>Jenya Ferapontov has pointed out that if  $b \neq 0$  the hydrodynamic reductions of (1.3) coincide with the hydrodynamic reductions of the dKP equation, despite the fact that dKP and (1.3) are not point or contact equivalent unless  $c = 0$ .

(+ + --) ASD Ricci flat metric with a conformal Killing vector whose self-dual derivative is null is of the form

$$g = e^{c\phi}(Vh - V^{-1}(d\phi + A)^2), \quad (1.4)$$

where  $\phi$  parametrises the orbits of the conformal Killing vector  $K = \partial/\partial\phi$  and

$$h = (dy + cudt)^2 - 4(dx + cudy - (cw + bu)dt)dt, \quad A = -\frac{1}{2}u_x dy + \left(\frac{c}{2}uu_x - u_y\right)dt, \quad V = \frac{1}{2}u_x.$$

This reduction explains the origin of the two parameters  $(b, c)$  in (1.3) as in this case the conformal symmetry is

$$K = c \times (\text{dilatation}) + b \times (\text{rotation with null SD derivative}).$$

## 2 Lax pair and $\text{Diff}(\Sigma^2)$ hierarchies

The system (1.3) admits a dispersionless Lax pair

$$L_0 = \frac{\partial}{\partial t} + (cw + bu - \lambda cu - \lambda^2) \frac{\partial}{\partial x} + b(w_x - \lambda u_x) \frac{\partial}{\partial \lambda}, \quad L_1 = \frac{\partial}{\partial y} - (cu + \lambda) \frac{\partial}{\partial x} - bu_x \frac{\partial}{\partial \lambda} \quad (2.1)$$

with a spectral parameter  $\lambda \in \mathbb{CP}^1$ . The overdetermined system of linear equations  $L_0\Phi = L_1\Phi = 0$ , where  $\Phi = \Phi(x, y, t, \lambda)$  admits solutions because equations (1.3) are equivalent to  $[L_0, L_1] = 0$ .

In general, consider the vector fields of the form

$$L_i = \frac{\partial}{\partial t_i} + A_i \frac{\partial}{\partial x} + B_i \frac{\partial}{\partial \lambda}, \quad (2.2)$$

where  $A_i, B_i$  are polynomials in  $\lambda$  with coefficients depending on  $(t^0 = x, t^i)$ . The flows of the  $\text{Diff}(\Sigma^2)$  hierarchy are defined by

$$[L_i, L_j] = 0, \quad i, j = 1, \dots, n. \quad (2.3)$$

To achieve a dual formulation, generalising Krichever's approach to dispersionless integrable systems [13], complexify the hierarchy (so that  $(t^0, t^i) \in \mathbb{C}^{n+1}$ ) and define a two-form  $\Omega$  on  $\mathbb{C}^{n+1} \times \mathbb{CP}^1$  by

$$\Omega(X, Y) = dt_1 \wedge \dots \wedge dt_n \wedge dx \wedge d\lambda(L_1, \dots, L_n, X, Y)$$

so that

$$\Omega = dx \wedge d\lambda + \sum_i (A_i d\lambda - B_i dx) \wedge dt_i + \sum_{i,j} (A_i B_j - B_j A_i) dt_i \wedge dt_j.$$

The two-form  $\Omega$  is simple and satisfies the Frobenius integrability conditions

$$\Omega \wedge \Omega = 0, \quad d\Omega = \Omega \wedge \beta$$

for some one-form  $\beta$ . We recover various dispersionless hierarchies as special cases of this formulation

- **SDiff( $\Sigma^2$ ) hierarchy.** The group  $\text{Diff}(\Sigma^2)$  reduces to  $\text{SDiff}(\Sigma^2)$  generated by Hamiltonian vector fields. The corresponding Lie algebra is homomorphic with the Poisson bracket algebra. The vector fields  $L_i$  preserve the two-form  $dx \wedge d\lambda$  and

$$A_i = \frac{\partial H_i}{\partial \lambda}, \quad B_i = -\frac{\partial H_i}{\partial x}.$$

The two-form is given by  $\Omega = dx \wedge d\lambda + \sum_i dH_i \wedge dt_i$  (where we have used (2.3)), and the  $\text{SDiff}(\Sigma^2)$  hierarchy is given by

$$\Omega \wedge \Omega = 0, \quad d\Omega = 0,$$

which is the original Krichever's formulation [13]. The Darboux theorem implies the existence of functions  $P, Q$  such that  $\Omega = dP \wedge dQ$ . These two functions are local coordinates on the twistor space, which is a quotient of  $\mathbb{C}^{n+1} \times \mathbb{CP}^1$  by the integrable distribution  $\{L_i\}$ . Both the dKP and  $\text{SU}(\infty)$  Toda hierarchies fit into this category [13, 23, 5].

- **Diff( $S^1$ ) hierarchy.** The group  $\text{Diff}(\Sigma^2)$  reduces to  $\text{Diff}(S^1)$ , where  $\Sigma^2 = TS^1$ . This case corresponds to

$$B_i = 0.$$

The underlying Lie algebra is that of a Wronskian with a Lie bracket  $\langle f, g \rangle = f_x g - f g_x$ . In this case  $\Omega = e \wedge d\lambda$ , where  $e = dx - \sum_i A_i dt_i$ . This one-form is integrable in the Frobenius sense

$$e \wedge de = 0,$$

where here  $d$  keeps  $\lambda = \text{const.}$  The twistor space fibres holomorphically over  $\mathbb{CP}^1$ . The hyper-CR hierarchy [4] and the universal hierarchy [17] are of this type.

### 3 Interpolating Einstein–Weyl structure

A three-dimensional Lorentzian Weyl structure  $(M, D, [h])$  consists of a 3-manifold  $M$ , a torsion-free connection  $D$  and a conformal metric  $[h]$  of Lorentzian signature such that the null geodesics of  $[h]$  are also geodesics for  $D$ . This condition is equivalent to

$$Dh = \omega \otimes h$$

for some one form  $\omega$ . Here  $h$  is a representative metric in the conformal class. If we change this representative by  $h \rightarrow \gamma^2 h$ , then  $\omega \rightarrow \omega + 2d \ln \gamma$ . A Weyl structure is called Einstein–Weyl if the conformally invariant equations

$$R_{(ab)} = \Lambda h_{ab}, \quad a, b, = 1, \dots, 3 \tag{3.1}$$

hold for some function  $\Lambda$ . Here  $R_{(ab)}$  is the symmetrised Ricci tensor of  $D$  and  $h_{ab}$  is a representative metric in a conformal class  $[h]$ . In practice the Einstein–Weyl structure is given by specifying the metric  $h \in [h]$ , and the one-form  $\omega$  which measures the difference between the Weyl connection  $D$  and the Levi–Civita connection of  $h$ .

The three-dimensional Einstein–Weyl condition is a dispersionless integrable system [5]: Let  $Z, W, \widetilde{W}$  be independent vector fields on  $M$  such that a contravariant metric in  $[h]$  is

$$h = h^{ab} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} = Z \otimes Z - 2(W \otimes \widetilde{W} + \widetilde{W} \otimes W).$$

Then there exists a connection  $D$  such that  $(M, [h], D)$  is Einstein–Weyl if the dispersionless Lax pair

$$L_0 = W - \lambda Z + f_0 \frac{\partial}{\partial \lambda}, \quad L_1 = Z - \lambda \widetilde{W} + f_1 \frac{\partial}{\partial \lambda}, \quad (3.2)$$

satisfies the integrability condition

$$[L_0, L_1] = 0 \quad \text{modulo} \quad L_0, L_1$$

for some functions  $(f_0, f_1)$  which are cubic polynomials in  $\lambda \in \mathbb{CP}^1$ . Conversely, every Einstein–Weyl structure arises from some Lax pair (3.2). The corresponding one form  $\omega$  can be read off from the Levi–Civita connection of  $h$  and the coefficients of  $(f_0, f_1)$ . This Lax formulation has a geometric origin which goes back to E. Cartan [2]

- Einstein–Weyl condition is equivalent to the existence of a two parameter family of totally geodesic null surfaces in  $M$ .

Comparing this with the Lax pair (2.1) for the interpolating system, and taking linear combinations to put (2.1) in the form (3.2) we find the corresponding Einstein–Weyl structure to be

$$\begin{aligned} h &= (dy + cudt)^2 - 4(dx + cudy - (cw + bu)dt)dt, \\ \omega &= -cu_x dy + (4bu_x + c^2uu_x - 2cu_y)dt. \end{aligned} \quad (3.3)$$

Taking the limits we recover various known Einstein–Weyl structures from (3.3). These structures can be characterised by the properties null shear-free and geodesic congruence  $dt$  (this elegant framework is described, in the Riemannian case, in [1])

- Setting  $b = 0, c = -1$  gives the hyper–CR Einstein–Weyl structure [4]. The congruence is divergence-free. This is a Lorentzian analogue Einstein–Weyl structures studied in [10, 1, 6].
- Setting  $c = 0, b = 1$  gives the dKP Einstein–Weyl spaces [5]. The congruence  $dt$  is now twist-free, and the dual vector  $\partial/\partial x$  is parallel with a weight  $-1/2$  with respect to the Weyl connection.

We remark that the Einstein–Weyl (3.3) structure can also be read off from the ASD Ricci–flat metric (1.4) using the Jones–Tod correspondence [12].

## 4 The Manakov–Santini system

The system (1.3) is a special case of the Manakov–Santini system [15, 14]

$$\begin{aligned} U_{xt} - U_{yy} + (UU_x)_x + V_x U_{xy} - V_y U_{xx} &= 0 \\ V_{xt} - V_{yy} + UV_{xx} + V_x V_{xy} - V_y V_{xx} &= 0, \end{aligned} \quad (4.1)$$

where  $U = U(x, y, t)$ ,  $V = V(x, y, t)$ . To see it, notice that the first equation in (1.3) implies the existence of  $v(x, y)$  such that  $u = v_x$ ,  $w = -v_y$ , and  $v$  satisfies

$$v_{xt} - v_{yy} + c(v_x v_{xy} - v_y v_{xx}) + b u v_{xx} = 0. \quad (4.2)$$

Differentiating the second equation in (1.3), and eliminating  $w$  yields

$$u_{xt} - u_{yy} - c(v_y u_{xx} - v_x u_{xy}) + b(u u_x)_x = 0. \quad (4.3)$$

Now assume the generic case when the constants  $c, b$  are non zero, and set  $U = bu$ ,  $V = cv$ . Then the system (4.2) and (4.3) is equivalent to (4.1) with an additional constraint

$$cU - bV_x = 0. \quad (4.4)$$

The Manakov–Santini system is more general than (1.3): Regarding the second equation in (4.1) as the definition of  $U$ , and substituting  $U$  to the first equation in (4.1) yields a fourth order scalar PDE for  $V$ . Thus the naive counting suggest that the general solution to (4.1) depends on 4 functions of 2 variables (a caution is needed as the resulting PDE is not in the Cauchy–Kowalewska form). In the special case when the constraint (4.4) holds both equations in (4.1) reduce to a single second order PDE for  $V$ . The solution depends on two functions of two variables, and a constant (the ratio  $b/c$ ).

The Manakov–Santini system also corresponds to an Einstein–Weyl structure

$$\begin{aligned} h &= (dy - V_x dt)^2 - 4(dx - (U - V_y)dt)dt, \\ \omega &= -V_{xx} dy + (4U_x - 2V_{xy} + V_x V_{xx})dt. \end{aligned} \quad (4.5)$$

To verify it set  $x^a = (y, x, t)$ . The (11), (12), (22), (23) components of the Einstein–Weyl equations hold identically. The (13) component vanishes if the second equation in (4.1) holds, and finally the (33) component vanishes if both equations in (4.1) are satisfied.

Conversely, consider the general conformal structure in  $(2 + 1)$  dimensions given in local coordinates  $x^a$  by a representative metric

$$h = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{pmatrix}.$$

Using the diffeomorphism freedom, and the conformal rescaling we can impose four constraint on six functions  $h_{ab}(x^c)$  as long as the resulting quadratic form is non-degenerate. We choose to set  $h_{11} = h_{12} = 0$ ,  $h_{13} = -2$ ,  $h_{23} = -A$ ,  $h_{33} = A^2 + 4B$ , where  $A$  and  $B$  are some functions of  $(x, y, t)$ , so that<sup>3</sup>

$$h = (dy - A dt)^2 - 4(dx - B dt)dt.$$

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<sup>3</sup>This is analogous to the existence of orthogonal coordinates in three dimensions. The proof is relatively straightforward in the real-analytic category, and more subtle in the smooth category.

Now given  $A, B$  we can always find two functions  $U, V$  such that  $A = V_x, B = U - V_y$  so that the metric is in the form (4.5)

Now we find the corresponding dual basis, and construct the Lax pair (3.2). Before imposing the integrability conditions it is convenient to take a linear combination of the vectors in this distribution, so that the resulting pair of vectors commutes exactly. This yields

$$L_0 = \partial_y - (\lambda + A)\partial_x + f_0\partial_\lambda, \quad L_1 = \partial_t - (\lambda^2 + \lambda A - B)\partial_x + f_1\partial_\lambda,$$

where the polynomials  $f_0$  and  $f_1$  are respectively cubic and quartic in  $\lambda$ . There is some additional freedom which will preserve the above form of the Lax pair. We can translate the fibres by  $\lambda \rightarrow \lambda + \kappa(x, y, t)$ . We use this freedom to set the linear term in  $f_0$  to zero. It is possible that some further coordinate freedom can be used to set the quadratic term in  $f_0$  to zero which would imply that  $\omega$  is given by (4.5). This would imply that every Einstein–Weyl structure is equivalent to the Manakov–Santini EW structure (4.5). So far we have been unable to find the right transformation, and we need to impose  $\partial_\lambda^2 f_0 = 0$  as an additional condition. Then the integrability condition  $[L_0, L_1] = 0$  implies (4.1).

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## Appendix. Reduction of the second heavenly equation

We shall prove that the system (1.3) arises as the most general symmetry reduction of the second heavenly equation (1.2) by a conformal Killing vector with null self-dual derivative.

Let  $\Theta = \Theta(W, Z, X, Y)$  satisfy the second heavenly equation and let the corresponding metric be given by (1.1). Let  $K$  be a conformal Killing vector for (1.1). Using Penrose’s two-component spinor formalism we can show that the conformal Killing equations and the Ricci identity imply that  $\nabla_{AA'}K^{A'}_B$  is covariantly constant, or otherwise  $g$  is of Petrov–Penrose type  $N$  and can be found explicitly. In the spin frame of the heavenly metric (1.1) the connection on the spin bundle  $\mathbb{S}$  vanishes, so  $\nabla_{A'A}K^{A'}_B$  is in fact constant. We are interested in the case where the self-dual derivative  $\phi_{AB} = \nabla_{A'(A}K^{A'}_{B)}$  is of rank one. Therefore we need to integrate the linear system

$$\nabla_{A'A}K^{A'}_B = \begin{pmatrix} 0 & c \\ -c & b \end{pmatrix}.$$

The constant  $c$  appears because  $K$  is a conformal Killing vector.

Following the method of Finley and Plebański [9] and using a freedom in the heavenly potential  $\Theta$  as well as in the choice of coordinates we find the general solution to be

$$K = (cZ + b)\partial_Z + (cX - 2bZ)\partial_X.$$

The conformal Killing equations  $\mathcal{L}_K g = cg$  now yield

$$\mathcal{L}_K(\Theta_{XX}) = -c\Theta_{XX}, \quad \mathcal{L}_K(\Theta_{XY}) = b, \quad \mathcal{L}_K(\Theta_{YY}) = c\Theta_{XX}.$$



Let  $U$  and  $T$  be functions such that  $K = \partial_T$  and  $\mathcal{L}_K(U) = 0$ . For  $c \neq 0$  we can take

$$T = \frac{\ln(cZ + b)}{c}, \quad U = \frac{2b}{c}T + \frac{2b^2 + Xc^2}{c^2(cZ + b)}.$$

The compatibility conditions for the Killing equations imply the existence of  $\tilde{G} = \tilde{G}(Y, W, U)$  such that

$$\Theta_{XX} = e^{-cT}\tilde{G}_{UU}, \quad \Theta_{XY} = \tilde{G}_{YU} + bT, \quad \Theta_{YY} = e^{cT}\tilde{G}_{YY}.$$

The heavenly equation (1.2) becomes

$$bU + \frac{2b}{c}\tilde{G}_{YU} + c(\tilde{G}_Y - U\tilde{G}_{YU}) + \tilde{G}_{UW} + \tilde{G}_{YY}\tilde{G}_{UU} - \tilde{G}_{YU}^2 = 0.$$

To obtain a simplified form define

$$G(Y, W, U) = \tilde{G}(Y, W, U) + \frac{b}{c}UY + \frac{b^2}{c^2}UW$$

so that

$$bU + c(G_Y - UG_{YU}) + G_{UW} + G_{YY}G_{UU} - G_{YU}^2 = 0. \quad (\text{A1})$$

Now rewrite (A1) in terms of differential forms

$$\begin{aligned} bU dY \wedge dU \wedge dW + c(G_Y dY \wedge dU \wedge dW - U dG_U \wedge dU \wedge dW) \\ + dG_U \wedge dY \wedge dU + dG_Y \wedge dG_U \wedge dW = 0. \end{aligned} \quad (\text{A2})$$

Define

$$x = G_U, \quad y = Y, \quad t = -W, \quad H(x, y, t) = xU(x, y, t) - G(Y, W, U(x, y, t)),$$

and perform a Legendre transform

$$\begin{aligned} dH &= d(xU - G) = Udx - G_Y dY - G_W dW \\ &= H_x dx + H_y dy + H_t dt. \end{aligned}$$

Therefore

$$U = H_x, \quad G_Y = -H_y, \quad G_W = H_t.$$

Differentiating these relations we find

$$G_{UU} = \frac{1}{H_{xx}}, \quad G_{YU} = -\frac{H_{xy}}{H_{xx}}, \quad G_{YY} = -H_{yy} + \frac{H_{xy}^2}{H_{xx}}.$$

The differential equation for  $H(x, y, t)$  is obtained from (A2)

$$H_{xt} + bH_x H_{xx} + c(H_{xy}H_x - H_y H_{xx}) = H_{yy}. \quad (\text{A3})$$

Setting  $u = H_x, w = -H_y$  we recover (1.4), where  $(u, w)$  solve (1.3).

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